Lecture Notes on Multiple Regression Analysis

When we came across more than two variables, which are known to be linearly related, we use multiple linear regression of one variable (dependent variable) on the remaining variables (independent variable). We restrict ourselves with three variables only and naturally we will have three linear regression equations.

Suppose (X_1, X_2, X_3) are three variables. First assume X_1 is dependent variable and X_2, X_3 are independent variables. We wish to obtain linear regression equation (plane) of X_1 on X_2 and X_3. Suppose

$$\mathbf{X_1 = a + b_{12.3} X_2 + b_{13.2} X_3} \qquad \text{....(A)}$$

be a linear regression equation (plane) of X_1 on X_2 and X_3. Here, $\mathbf{a, b_{12.3}, b_{13.2}}$ are called parameters of the regression equation (plane). Yule's notations:

$\mathbf{b_{12.3}}$**:** is called regression coefficient of X_1 on X_2 for fixed value of X_3

$\mathbf{b_{13.2}}$ **:** is called regression coefficient of X_1 on X_3 for fixed value of $X_{2.}$

Since the parameters $\mathbf{a, b_{12.3}, b_{13.2}}$ unknown, we need to estimate these by using suitable statistical technique for given set of observations on (X_1, X_2, X_3). In the following, we discuss least square method to estimate these parameter.

Least Square Method to obtain Regression Equation:

Suppose $\mathbf{X_1 = a + b_{12.3} X_2 + b_{13.2} X_3}$ be a linear regression equation (plane) of X_1 on X_2 and X_3 and (X_{1j}, X_{2j}, X_{3j}), j = 1, 2, ..., N be N observations on (X_1, X_2, X_3). We have to estimate $\mathbf{a, b_{12.3}, b_{13.2}}$ such that sum of the squares of errors in observed values of $\mathbf{X_1}$ ($\mathbf{X_{1j}}$) and estimated values of $\mathbf{X_1}$ based on X_{2j}, X_{3j} ($\mathbf{a + b_{12.3} X_{2j} + b_{13.2} X_{3j} = e_{1.23}}$ **(**say)) is least. That is, we have to obtain $\mathbf{a, b_{12.3}, b_{13.2}}$ such that $\mathbf{S(a, b_{12.3}, b_{13.2})}$ is minimum with respect to $\mathbf{a, b_{12.3}, b_{13.2}}$, where

$$S(a, b_{12.3}, b_{13.2}) = \sum_{j=1}^{N}(X_{1j} - e_{1.23,j})^2 = \sum_{j=1}^{N}(X_{1j} - a - b_{12.3}X_{2j} - b_{13.2}X_{3j})^2.$$

By using the principle of maxima and minima on $\mathbf{S(a, b_{12.3}, b_{13.2})}$, we have to estimate $\mathbf{a, b_{12.3}, b_{13.2}}$. Now, differentiating $\mathbf{S(a, b_{12.3}, b_{13.2})}$ with respect to $\mathbf{a, b_{12.3}, b_{13.2}}$ and equating these differentiations to zero we get following three respective normal equations:

$$Na + \left(\sum_{j=1}^{N} X_{2j}\right)b_{12.3} + \left(\sum_{j=1}^{N} X_{3j}\right)b_{13.2} = \left(\sum_{j=1}^{N} X_{1j}\right) \quad \text{...(I)}$$

$$\left(\sum_{j=1}^{N} X_{2j}\right)a + \left(\sum_{j=1}^{N} X_{2j}^2\right)b_{12.3} + \left(\sum_{j=1}^{N} X_{2j}X_{3j}\right)b_{13.2} = \left(\sum_{j=1}^{N} X_{2j}X_{1j}\right) \quad \text{...(II)}$$

$$\left(\sum_{j=1}^{N} X_{3j}\right)a + \left(\sum_{j=1}^{N} X_{3j}X_{2j}\right)b_{12.3} + \left(\sum_{j=1}^{N} X_{3j}^2\right)b_{13.2} = \left(\sum_{j=1}^{N} X_{3j}X_{1j}\right) \quad \text{...(III)}$$

We have to solve these normal equations simultaneously for **a, $b_{12.3}$, $b_{13.2}$.**

N×(II) – $\left(\sum_{j=1}^{N} X_{2j}\right)$× (I) gives:

$$\left\{N\left(\sum_{j=1}^{N} X_{2j}^2\right) - \left(\sum_{j=1}^{N} X_{2j}\right)^2\right\} b_{12.3} + \left\{N\left(\sum_{j=1}^{N} X_{2j}X_{3j}\right) - \left(\sum_{j=1}^{N} X_{2j}\right)\left(\sum_{j=1}^{N} X_{3j}\right)\right\} b_{13.2} = N\left(\sum_{j=1}^{N} X_{2j}X_{1j}\right) - \left(\sum_{j=1}^{N} X_{2j}\right)\left(\sum_{j=1}^{N} X_{1j}\right) \quad \text{...(IV)}$$

N×(III) – $\left(\sum_{j=1}^{N} X_{3j}\right)$× (I) gives:

$$\left\{N\left(\sum_{j=1}^{N} X_{3j}X_{2j}\right) - \left(\sum_{j=1}^{N} X_{3j}\right)\left(\sum_{j=1}^{N} X_{2j}\right)\right\} b_{12.3} + \left\{N\left(\sum_{j=1}^{N} X_{3j}^2\right) - \left(\sum_{j=1}^{N} X_{3j}\right)^2\right\} b_{13.2} = N\left(\sum_{j=1}^{N} X_{3j}X_{1j}\right) - \left(\sum_{j=1}^{N} X_{3j}\right)\left(\sum_{j=1}^{N} X_{1j}\right) \quad \text{...(V)}$$

Dividing (IV) and (V) by N^2 and using the definition of co-variance between X_m and X_n (say S_{mn}) as $s_{mn} = Cov(X_m, S_n) = \frac{1}{N}\sum_{j=1}^{N} X_{mj}X_{nj} - \bar{X}_m\bar{X}_n$, we get following two equations:

$$\mathbf{s_{22}\, b_{12.3} + s_{23}\, b_{13.2} = s_{21} \quad \text{and} \quad s_{32}\, b_{12.3} + s_{33}\, b_{13.2} = s_{31}} \quad \textbf{...(VI)}$$

In matrix form, these equations becomes:

$$\begin{bmatrix} s_{22} & s_{23} \\ s_{32} & s_{33} \end{bmatrix} \times \begin{bmatrix} b_{12.3} \\ b_{13.2} \end{bmatrix} = \begin{bmatrix} s_{21} \\ s_{31} \end{bmatrix}$$

Let,

$$S = \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix}$$: a variance-covariance matrix of (X_1, X_2, X_3). Hence,

$b_{12.3} = \frac{\begin{vmatrix} s_{21} & s_{23} \\ s_{31} & s_{33} \end{vmatrix}}{\begin{vmatrix} s_{22} & s_{23} \\ s_{32} & s_{33} \end{vmatrix}} = \frac{-S_{12}}{S_{11}}$ and $b_{13.2} = \frac{\begin{vmatrix} s_{22} & s_{21} \\ s_{32} & s_{31} \end{vmatrix}}{\begin{vmatrix} s_{22} & s_{23} \\ s_{32} & s_{33} \end{vmatrix}} = \frac{-S_{13}}{S_{11}}$, where $\mathbf{S_{mn}}$= co-factor of the element $\mathbf{s_{mn}}$ = $(-1)^{(m+n)}$× Minor of the element s_{mn} in matrix S.

We know that $s_{mn} = Cov(X_m , X_n) = r_{mn}\sqrt{s_{mm} \times s_{nn}} = r_{mn} \times \sigma_m \times \sigma_n$, where σ_m is a standard deviation of X_m.

Therefore, if we define = $\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$: a correlation matrix of (X_1, X_2, X_3) then

$$b_{12.3} = \frac{\begin{vmatrix} s_{21} & s_{23} \\ s_{31} & s_{33} \end{vmatrix}}{\begin{vmatrix} s_{22} & s_{23} \\ s_{32} & s_{33} \end{vmatrix}} = \frac{\begin{vmatrix} r_{21}\times\sigma_2\times\sigma_1 & r_{23}\times\sigma_2\times\sigma_3 \\ r_{31}\times\sigma_3\times\sigma_1 & r_{33}\times\sigma_3\times\sigma_3 \end{vmatrix}}{\begin{vmatrix} r_{22}\times\sigma_2\times\sigma_2 & r_{23}\times\sigma_2\times\sigma_3 \\ r_{32}\times\sigma_3\times\sigma_2 & r_{33}\times\sigma_3\times\sigma_3 \end{vmatrix}} = \frac{\sigma_1\times\sigma_2\times\sigma_3{}^2\begin{vmatrix} r_{21} & r_{23} \\ r_{31} & r_{33} \end{vmatrix}}{\sigma_2{}^2\times\sigma_3{}^2\begin{vmatrix} r_{22} & r_{23} \\ r_{32} & r_{33} \end{vmatrix}} = \frac{-\sigma_1\times R_{12}}{\sigma_2\times R_{11}} \quad \text{(VII) and}$$

$$b_{13.2} = \frac{\begin{vmatrix} s_{22} & s_{21} \\ s_{32} & s_{31} \end{vmatrix}}{\begin{vmatrix} s_{22} & s_{23} \\ s_{32} & s_{33} \end{vmatrix}} = \frac{\begin{vmatrix} r_{22}\times\sigma_2\times\sigma_2 & r_{21}\times\sigma_2\times\sigma_1 \\ r_{32}\times\sigma_3\times\sigma_2 & r_{31}\times\sigma_3\times\sigma_1 \end{vmatrix}}{\begin{vmatrix} r_{22}\times\sigma_2\times\sigma_2 & r_{23}\times\sigma_2\times\sigma_3 \\ r_{32}\times\sigma_3\times\sigma_2 & r_{33}\times\sigma_3\times\sigma_3 \end{vmatrix}} = \frac{\sigma_1\times\sigma_2{}^2\times\sigma_3\begin{vmatrix} r_{22} & r_{21} \\ r_{31} & r_{33} \end{vmatrix}}{\sigma_2{}^2\times\sigma_3{}^2\begin{vmatrix} r_{22} & r_{23} \\ r_{32} & r_{33} \end{vmatrix}} = \frac{-\sigma_1\times R_{13}}{\sigma_3\times R_{11}} \text{(VIII)}.$$

Substituting $\mathbf{b_{12.3}}$, $\mathbf{b_{13.2}}$ in (I) we get $a = \bar{X}_1 - b_{12.3}\bar{X}_2 - b_{13.2}\bar{X}_3$ and hence from (A) the linear regression equation (plane) of X_1 on X_2 and X_3 becomes: $X_1 - \bar{X}_1 = b_{12.3}(X_2 - \bar{X}_2) + b_{13.2}(X_3 - \bar{X}_3)$.