Properties of Multiple Regression Coefficients, Multiple Correlation Coefficients and Partial Correlation Coefficients

We have: $\boldsymbol{R_{1.23}} = \left(\mathbf{1} - \frac{|\boldsymbol{R}|}{\boldsymbol{R_{11}}}\right)^{1/2}$

Therefore, $\mathbf{1} - \boldsymbol{R^2_{1.23}} = \frac{|\boldsymbol{R}|}{\boldsymbol{R_{11}}}$

Here, $|\boldsymbol{R}| = (\mathbf{1} - \boldsymbol{r^2_{23}}) - \boldsymbol{r_{12}}(\boldsymbol{r_{12}} - \boldsymbol{r_{13}}\boldsymbol{r_{23}}) + \boldsymbol{r_{13}}(\boldsymbol{r_{12}}\boldsymbol{r_{23}} - \boldsymbol{r_{13}})$

$$= \mathbf{1} - \boldsymbol{r^2_{12}} - \boldsymbol{r^2_{13}} - \boldsymbol{r^2_{23}} + \mathbf{2}\boldsymbol{r_{12}}\boldsymbol{r_{13}}\boldsymbol{r_{23}}$$

$$|\boldsymbol{R_{11}}| = (\mathbf{1} - \boldsymbol{r^2_{23}})$$

$$\mathbf{1} - \boldsymbol{R^2_{1.23}} = \frac{1 - r^2_{12} - r^2_{13} - r^2_{23} + 2r_{12}r_{13}r_{23}}{(1 - r^2_{23})} \qquad \textbf{... (1)}$$

Also, we have: $\boldsymbol{r^2_{13.2}} = \frac{\boldsymbol{R^2_{13}}}{\boldsymbol{R_{11}}\boldsymbol{R_{33}}} = \frac{(r_{12}r_{23} - r_{13})^2}{(1 - r^2_{23})(1 - r^2_{12})}$

Therefore, $\mathbf{1} - \boldsymbol{r^2_{13.2}} = \mathbf{1} - \frac{(r_{12}r_{23} - r_{13})^2}{(1 - r^2_{23})(1 - r^2_{12})} = \frac{1 - r^2_{12} - r^2_{13} - r^2_{23} + 2r_{12}r_{13}r_{23}}{(1 - r^2_{23})(1 - r^2_{12})}$ **...(2)**

From (1) and (2), $-\boldsymbol{r^2_{13.2}} = \frac{1 - R^2_{1.23}}{(1 - r^2_{12})}$. That is,

$$\mathbf{1} - \boldsymbol{R^2_{1.23}} = (\mathbf{1} - \boldsymbol{r^2_{12}})(\mathbf{1} - \boldsymbol{r^2_{13.2}}) \qquad \textbf{...(3)}$$

Similarly,

$$\mathbf{1} - \boldsymbol{R^2_{1.23}} = (\mathbf{1} - \boldsymbol{r^2_{13}})(\mathbf{1} - \boldsymbol{r^2_{12.3}})$$

$$\mathbf{1} - \boldsymbol{R^2_{2.13}} = (\mathbf{1} - \boldsymbol{r^2_{21}})(\mathbf{1} - \boldsymbol{r^2_{23.1}})$$

$$\mathbf{1} - \boldsymbol{R^2_{2.13}} = (\mathbf{1} - \boldsymbol{r^2_{23}})(\mathbf{1} - \boldsymbol{r^2_{21.3}})$$

$$\mathbf{1} - \boldsymbol{R^2_{3.12}} = (\mathbf{1} - \boldsymbol{r^2_{31}})(\mathbf{1} - \boldsymbol{r^2_{32.1}})$$

$$\mathbf{1} - \boldsymbol{R^2_{3.12}} = (\mathbf{1} - \boldsymbol{r^2_{32}})(\mathbf{1} - \boldsymbol{r^2_{31.2}})$$

Therefore, $\mathbf{1} - \boldsymbol{R^2_{1.23}} \leq (\mathbf{1} - \boldsymbol{r^2_{12}})$ and $\mathbf{1} - \boldsymbol{R^2_{1.23}} \leq (\mathbf{1} - \boldsymbol{r^2_{13}})$.

Since, $0 \leq (\mathbf{1} - \boldsymbol{r^2_{13.2}}) \leq \mathbf{1}$ and $0 \leq (\mathbf{1} - \boldsymbol{r^2_{12.3}}) \leq \mathbf{1}$

That is, $\boldsymbol{R^2_{1.23}} \geq \boldsymbol{r^2_{12}}$ and $\boldsymbol{R^2_{1.23}} \geq \boldsymbol{r^2_{13}}$ **OR** $R_{1.23} \geq |r_{12}|$ and $R_{1.23} \geq |r_{13}|$. (Since $R_{1.23} \geq 0$)

$$\mathbf{R_{1.23} \geq Maximum\ \{|r_{12}|, |r_{13}|\}} \qquad \textbf{...(4)}$$

Similarly,

$$\mathbf{R_{2.13} \geq Maximum\ \{|r_{21}|, |r_{23}|\}}$$

$$\mathbf{R_{3.12} \geq Maximum\ \{|r_{31}|, |r_{32}|\}}$$

Partial Regression Coefficients in terms of Simple Regression Coefficients:

We have,

$b_{12.3} = \frac{-\sigma_1 \times R_{12}}{\sigma_2 \times R_{11}} = \frac{\sigma_1(r_{12}-r_{13}r_{23})}{\sigma_2\left(1-r_{23}^2\right)} = \frac{\sigma_1(r_{12}-r_{13}r_{32})}{\sigma_2(1-r_{23}r_{32})}$ (since $r_{ij} = r_{ji}$)

$= \frac{\sigma_1(\frac{\sigma_2}{\sigma_1}b_{12}-\frac{\sigma_3}{\sigma_1}b_{13}\frac{\sigma_2}{\sigma_3}b_{32})}{\sigma_2(1-\frac{\sigma_3}{\sigma_2}b_{23}\frac{\sigma_2}{\sigma_3}b_{32})}$ (since $b_{ij} = r_{ij}\frac{\sigma_i}{\sigma_j}$; $r_{ij} = b_{ij}\frac{\sigma_j}{\sigma_i}$ or $r_{ij} = b_{ji}\frac{\sigma_i}{\sigma_j}$, for all i, j = 1, 2, 3)

$= \frac{(b_{12}-b_{13}b_{32})}{(1-b_{23}b_{32})}$.

Similarly,

$b_{13.2} = \frac{(b_{13}-b_{12}b_{23})}{(1-b_{32}b_{23})}$ and $b_{23.1} = \frac{(b_{23}-b_{21}b_{12})}{(1-b_{31}b_{13})}$ etc.